Lecture 16. Bases of vector spaces

Def A basis of a vector space V is a set of linearly independent vectors which span V.

e.g. the standard basis of $\mathbb{R}^n$ given by $\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n$.

Thm If a vector space V is spanned by a finite set S, it has a basis given by a subset of S.

Note In fact, we can find a basis of V as follows:

- If S consists of linearly independent vectors, it is a basis.
- Otherwise, we remove from S a vector which is a linear combination of the others and start over.

Prop Let A be a matrix.

(1) If the solution of the equation $A\vec{x} = \vec{0}$ is parametrized by

$$\vec{x} = t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_d\vec{v}_d \quad (d = \#\text{ of free variables})$$

a basis of $\mathrm{Nul}(A)$ is given by $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_d$.

(2) A basis of $\mathrm{Col}(A)$ is given by the columns that contain a position of a leading 1 in $\mathrm{RREF}(A)$.

(3) A basis of $\mathrm{Row}(A)$ is given by the nonzero rows in $\mathrm{RREF}(A)$.

Note The identities $\mathrm{Col}(A) = \mathrm{Row}(A^T)$ and $\mathrm{Row}(A) = \mathrm{Col}(A^T)$ yield

- a basis of $\mathrm{Col}(A)$ given by the nonzero rows in $\mathrm{RREF}(A^T)$,
- a basis of $\mathrm{Row}(A)$ given by the columns of A^T that contain a position of a leading 1 in $\mathrm{RREF}(A^T)$.

Prop Let A be an $m\times n$ matrix with columns $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$.

(1) $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linearly independent

$\Longleftrightarrow$ RREF(A) has a leading 1 in every column

(2) $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ span $\mathbb{R}^m$

$\Longleftrightarrow$ RREF(A) has a leading 1 in every row

(3) $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ form a basis of $\mathbb{R}^m$

$\Longleftrightarrow$ $\text{RREF}(A) = I$ $\Longleftrightarrow$ A is a square matrix with $\det(A) \neq 0$

pf (1) This was proved in Lecture 8.

(2) $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ span $\mathbb{R}^m$

$\Longleftrightarrow$ Every vector in $\mathbb{R}^m$ is a linear combination of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$

$\Longleftrightarrow$ $A\vec{x} = \vec{v}$ has a solution for any $\vec{v} \in \mathbb{R}^m$

$\Longleftrightarrow$ The linear transformation with standard matrix A is surjective

$\Longleftrightarrow$ RREF(A) has a leading 1 in every row

(3) $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ form a basis of $\mathbb{R}^m$

$\Longleftrightarrow$ $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linearly independent and span $\mathbb{R}^m$

$\Longleftrightarrow$ RREF(A) has a leading 1 in every column and row

$\Longleftrightarrow$ $\text{RREF}(A) = I$

$\Longleftrightarrow$ A is a square matrix with an inverse

$\Longleftrightarrow$ A is a square matrix with $\det(A) \neq 0$

Ex Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & -4 & 2 \\ 4 & 3 & -6 & 8 \\ 3 & 2 & -5 & 5 \end{bmatrix} \quad \text{with} \quad \mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(1) Find a basis of Nul(A).

Sol We solve the equation $A\vec{x} = \vec{0}$ using RREF(A).

$$\begin{cases} x_1 \quad\quad -3x_3 - x_4 = 0 \\ x_2 + 2x_3 + 4x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 3x_3 + x_4 \\ x_2 = -2x_3 - 4x_4 \end{cases}$$

Set $x_3 = s$ and $x_4 = t$ (free variables)

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s + t \\ -2s - 4t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow$ A basis of Nul(A) is given by $\boxed{\begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}}$

(2) Find a basis of Col(A).

Sol RREF(A) has leading 1's in column 1 and column 2.

$\Rightarrow$ A basis of Col(A) is given by $\boxed{\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}}$ col 1, col 2 in A

(3) Find a basis of Row(A).

Sol The nonzero rows in RREF(A) are row 1 and row 2.

$\Rightarrow$ A basis of Row(A) is given by $\boxed{\begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}}$ row 1, row 2 in RREF(A)

Ex Determine whether each set of vectors is a basis of $\mathbb{R}^3$

(1) $\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

Sol $A = \begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 0 & 1 \end{bmatrix}$ is not a square matrix.

$\Rightarrow$ The set is $\boxed{\text{not a basis of } \mathbb{R}^3}$

(2) $\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$

Sol $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -2 & 5 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow \det(A) = 10 \neq 0$

$\Rightarrow$ The set is $\boxed{\text{a basis of } \mathbb{R}^3}$

Note We can get the same answer by computing RREF(A)

(3) $\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$

Sol $A = \begin{bmatrix} 2 & 3 & 5 \\ 4 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \det(A) = 0$

$\Rightarrow$ The set is $\boxed{\text{not a basis of } \mathbb{R}^3}$

Note We can get the same answer by computing RREF(A)